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An alternative dispersion equation for water waves over an inclined bed

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A classical approach to extending the validity of Airy's dispersion relation for surface gravity waves by Friedrichs (1948) to gentle slopes (of special inclinations) is here re-examined with extended small-slope asymptotics using the full linear harmonic function theory combined with the method of steepest descent. A new dispersion relation emerges that appears to give significantly increased accuracy over sloping beds when tested on the plane beach problem with various forms of the mild-slope equation (MSE) and global error reductions of the order 50% are noted in some 'from deep to shore' computations. Unlike the classical formula, the new formula predicts a discontinuous wavenumber at a place where the bottom slope is discontinuous. Preliminary tests examining the reflection coefficient with the basic (early version) MSE over ramp-type profiles indicate that this is not a major problem and numerical results using wavenumber calculated by the new dispersion relation are qualitatively similar to those of the modified MSE (MMSE) developed in Chamberlain & Porter (1995). When the new formula is applied (with mass conservation) to the MMSE on the ramp, results are almost identical to those of a full linear model for inclines having a gradient up to 8:1.

It is also shown that the dominant asymptotic analysis, responsible for the new formula, is valid for all slope angles $\alpha < \pi/2$ and not just the special angles considered by Friedrichs.

1. Introduction

The theory of gravity waves within the framework of a small-amplitude linear perfect-fluid theory over non-horizontal bottoms is often treated, for numerical expediency, by means of one or other of the various forms of mild slope equations (MSE) that now exist (see e.g. Berkhoff 1973; Booij 1983; Massel 1993; Chamberlain & Porter 1995; Porter & Staziker 1995; or most recently Porter 2003; Kim & Kwang 2004). All these rely, in one way or another, on the classical dispersion relation for gravity waves on a horizontal bed at depth h usually written in the form

$$k \tanh kh = \omega^2/g. \tag{1.1}$$

Here k is a wavenumber and ω the circular frequency of the monochromatic wave. By non-dimensionalizing length with respect to g/ω^2 the alternative more compact form $k = \coth kh$ is obtained and solutions are readily generated for given h by fixed point iteration. Its use on gentle inclines was first proposed by Burnside (1914) who justified it through energy flux arguments on the assumption that reflection was insignificant.

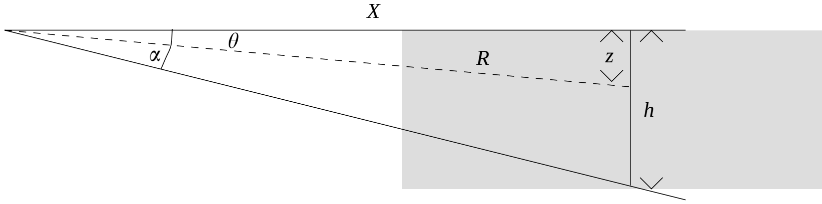

FIGURE 1. Schematic representation of beach and shelving region as used in Friedrichs' (1948) model; α is the sloping beach angle and the shaded region recedes to infinity as $\alpha \to 0$. Note that $X\alpha \sim h$ is held constant and $X\theta \sim z$.

Friedrichs (1948) extended the justification of Burnside (1914) by examining the potential function asymptotically for small beach slopes and this extension is perhaps that adopted by others as justification for using the relation locally even when h varies. The approach in Friedrichs (1948) is to assume the distance of the observation point from a shoreline origin is $O(\alpha^{-1})$ where α is the angle of inclination of a plane bed to the horizontal (see figure 1).

There are four essential points arising in Friedrichs' work which will be addressed here:

(i) the theory was developed only for special slope angles $\alpha = \pi/2N$ where N is integer;

(ii) the asymptotic theory was restricted to the surface behaviour of the potential;

(iii) only the dominant term was considered in an asymptotic expansion;

(iv) only qualitative discussion was given on the small depth limitations of the dispersion relation.

The main purpose of the present paper is to examine these issues in more detail and in particular it will be found that, through extension of the asymptotic analysis (here using the Mellin transform solution first derived by the author in Ehrenmark 1987), a new dispersion relation emerges which is also dependent upon bottom gradient.

Thus, that Mellin transform model is briefly outlined in the next section, following which, in § 3, the relevant results of Friedrichs' are re-derived in the revised model and, in particular, the conventional Airy dispersion relation is seen to emerge at this lowest order of consideration. The shoaling coefficient also emerges as the anticipated multiple of the group velocity expression. In § 4, the analysis is extended to that of a two-term asymptotic expansion using the method of steepest descent and here a new dispersion relation emerges from the saddle point condition. This is the main result of the present work. It will be found that the conventional Airy relation is replaced by the modified dispersion equation

$$k = \coth((\alpha \cot \alpha)kh)$$

to this order of approximation. Thus the Airy equation $k = \coth kh$ is recovered in the limit $\alpha \to 0$. That the value of k defined by the new relation truly measures wavenumber is substantiated (in Appendix B) by differentiation of the argument of the oscillatory component of the potential.

In § 5 are presented the results of two types of tests on this new formula. First the plane beach problem is solved using two different forms of the MSE and errors calculated from the known exact solutions. These tests all confirm that the new wavenumber values agree better than do the classical values with the 'benchmark' values developed in Ehrenmark & Williams (2001) using the exact solution of the problem. These latter values had previously been shown to produce a much improved response of the various MSEs for the beach problem. The actual performance of the new wavenumbers (allowing also for revised group velocity values) are seen to give significantly reduced errors – usually as much as 40–50% reduction in the global error when the potential is computed from deep to almost the shoreline.

The second set of tests considered involves the ramp profile designed by Booij (1983). This has been used extensively by other authors, e.g. Chamberlain & Porter (1995) or Kim & Kwang (2004), to test the response of the reflection coefficients to varying slopes of the ramp and has become a benchmark test for many MSE and similar approximations. Results are considered for both mass conservation and surface slope continuity models and the improvement found by using the revised wavenumber calculation is again most significant. For example the accuracy when working with the basic MSE using the new k-values is qualitatively similar to that of the modified MSE using old values and this is found to be the case even for profiles having a gradient as large as 8. Finally, a test on the modified MSE (Porter & Staziker 1995) (with mass conservation) also shows a significant improvement despite the values in Porter & Staziker (1995) being already very accurate. The new values are almost indistinguishable from those calculated by a two-dimensional linear model.

Suggestions for further testing are outlined in the concluding § 6 along with some numerical assessment of the small depth limitations of the dispersion relation on a steep incline. The consideration of extending the theory, uniformly to beaches of all angles, turns out to be mainly mathematical (possibly of limited interest to users) and is therefore presented as Appendix A.

2. A model of waves on a plane incline

The search of an extension to equation (1.1) will be based on the approach adopted by Friedrichs whereby the integrand of a contour integral expression for the velocity potential is first expanded (in this case for asymptotically small values of beach slope) and the path then distorted to pass through saddle points of the dominant part of the integrand. The anticipation that the solution should have a well-known structure in the far field, which needs to be recovered from the integral expression, then allows the condition at the saddle point to generate the subsequent dispersion relation.

We begin by writing, as an inverse Mellin transform, the potential function Φ for the regular standing wave of the classical scattering problem on a plane beach. For time-harmonic motion of circular frequency ω, such an expression satisfying $\nabla\Phi = 0$ in the fluid domain and the bottom and surface conditions

$$\frac{\partial\Phi}{\partial\theta} = 0 \text{ on } \theta = -\alpha, \tag{2.1}$$

$$\frac{\partial\Phi}{\partial\theta} = R\Phi \text{ on } \theta = 0, \tag{2.2}$$

can be written formally as $\Phi = \mathrm{Re}\{\phi(R,\theta)\mathrm{e}^{\mathrm{i}\omega t}\}$ where,

$$\phi = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \Gamma(s)R^{-s}B(s)\sin s\pi\frac{\cos s(\theta+\alpha)}{\cos s\alpha}\mathrm{d}s. \tag{2.3}$$

Here R, θ are circular polar coordinates such that $\theta = 0$ represents the still water line, $\theta = -\alpha$ the bottom, $R = 0$ the shoreline and the contour in the integral is indented to the right at the origin (see e.g. Ehrenmark 1996). Note that lengths have been non-dimensionalized through g/ω^2. This wave has amplitude $(2\pi)^{1/2}$ at infinity. The equation

$$B(s+1) = B(s)\tan s\alpha \tag{2.4}$$

arises directly from application of the free-surface boundary condition. If we assume the 'very simple' beach angles $\alpha = \pi/2N$, for integer N, then

$$B(s)\sin\pi s = (2\pi)^{1/2}2^{N-1}\prod_{j=0}^{N-1}\cos(s+j)\alpha, \ 0 < \mathrm{Re}\,(s) < 1. \tag{2.5}$$

Note however, that $B(s+1) = B(s)\tan s\alpha$ (thus enabling analytic continuation of (2.5) into $\mathrm{Re}\,(s) < 0$) and that

$$B(s)B(1-s) = \frac{\pi}{\sin\pi s}$$

(incidentally, this is valid regardless of the nature of α – see Appendix A). This facilitates an immediate asymptotic expansion of $|B(\mathrm{i}\tau)|$ for small α which is uniformly valid if $|\tau| = o(N)$. Because $|B(\mathrm{i}\tau)|^2 = (\pi\coth(\pi\tau/2N)/(\sinh\pi\tau)$, it follows that

$$|B(\mathrm{i}\tau)| \sim \left(\frac{2N}{\tau\sinh\pi\tau}\right)^{1/2} + O\left(\alpha^{3/2}\right), \ \alpha \to 0.$$

The idea of Friedrichs was to deform the integration contour to pass through saddle points of the integrand and, essentially, (1.1) arose from the saddle point method. The resulting expression for wave potential is then valid only asymptotically near the surface (see p. 112 of Friedrichs 1948).

Here it is intended to use the ansatz given above (equations (2.3) and (2.5)) in a similar way but one which yields the anticipated behaviour in the entire water column. With the benefit of hindsight, one would put

$$s = \mathrm{i}k\zeta X$$

and look for a value of k which yields a saddle point at $\zeta = 1$. In this way we can recover, from the dominant term obtained by the saddle point method, the depth variation

$$\frac{\cosh k(z+h)}{\cosh kh},$$

given that, asymptotically, $X|\theta| \sim z$, $X\alpha \sim h$ as $\alpha \to 0$.

3. Saddle point method

Initially it is assumed that $\alpha = \pi/2N$ where N is integer. It may be anticipated that the integrand needs to be estimated at (or near) $s = \mathrm{i}Y$ for arbitrarily large Y. Specifically, $R = X(1 + z^2/2X^2 + O(\alpha^4))$ and if $s = \mathrm{i}k\eta X$, then with the help of Stirling's formula (Spain & Smith 1968, p. 96) it follows that

$$R^{-s}\Gamma(s) = \sqrt{(2\pi/X)}\exp\left(\left(\mathrm{i}k\eta X - \tfrac{1}{2}\right)\log \mathrm{i}k\eta - \mathrm{i}k\eta X + \frac{1}{\mathrm{i}X}\left[\frac{1}{12k\eta} + \frac{k\eta z^2}{2}\right] + \mathrm{O}(X^{-2})\right) \tag{3.1}$$

and with the help of the Euler–Maclaurin sum formula (Fröberg 1985, p. 305) in equation (2.5) it also follows that

$$\frac{B(s)\sin \pi s}{\sqrt{(2\pi)2^{N-1}}}$$
$$= \exp\left\{\frac{1}{\alpha}\int_0^{\pi/2} \log\cos(\sigma + \mathrm{i}k\eta h)\mathrm{d}\sigma + \tfrac{1}{2}\log\coth(kh\eta) + \frac{\pi\mathrm{i}}{4} - \frac{\mathrm{i}\alpha}{6\sinh 2kh\eta} + O(\alpha^3)\right\}.$$

Accordingly, if the potential is expressed in the form

$$\phi = \frac{kh}{2\alpha}\int_{-\infty}^{\infty} \mathrm{e}^{Xw(\eta)}\Phi(X,\eta)\frac{\cosh k\eta(z+h)}{\cosh k\eta h}\mathrm{d}\eta$$

with the contour now suitably indented at the origin (below the axis), then

$$w(\eta) = \mathrm{i}k\eta \log\frac{\mathrm{i}k\eta}{\mathrm{e}} + \frac{\pi}{2h}\log 2 + \frac{1}{h}\int_0^{\pi/2} \log\cos(\sigma + \mathrm{i}k\eta h)\,\mathrm{d}\sigma \tag{3.2}$$

and

$$\Phi = (\mathrm{i}k\eta X)^{-1/2}\exp\left\{\frac{1}{6\mathrm{i}X}\left(\frac{1}{2k\eta} + \frac{h}{\sinh 2kh\eta} + 3kz^2\eta\right) + \frac{\pi\mathrm{i}}{4} + \tfrac{1}{2}\log\coth kh\eta + O(\alpha^3)\right\}. \tag{3.3}$$

3.1. *Friedrichs' result*

Some of the analysis of Friedrichs (1948) is revisited here. Of pivotal interest is the conclusion therein that (1.1) could be used on gentle inclines. This was based on the observation that when the asymptotic wave form was found to be $-2A(\lambda)\sin\{\alpha^{-1}\kappa(\lambda) + \pi/4\}$ (where A is an amplitude function), the local wavelength Λ determined from $\Lambda = 2\pi\mathrm{d}\alpha X/\mathrm{d}\kappa = 2\pi\lambda$ was such that when the local depth was h_0 then

$$h_0 = \lambda_0 \tanh^{-1}\lambda_0,$$

approximately, λ_0 also being a local value.

In the present work, from equation (3.2) $w'(\eta) = \mathrm{i}k\log(k\eta\tanh k\eta h)$ so that a saddle point will exist at $\eta = \pm 1$ if $k = \coth kh$, which is the non-dimensional form of the usual dispersion relation when k represents the (local) wavenumber. The value of $w(1)$ is required. After transforming the integral in equation (3.2)

$$w(1) = \mathrm{i}k\log\frac{\mathrm{i}k}{\mathrm{e}} - \frac{\mathrm{i}\pi^2}{8h} + \frac{k\pi}{2} + \frac{\mathrm{i}}{2h}\int_{-1}^{1}\log(1+\lambda^{-1}u)\frac{\mathrm{d}u}{u}\quad \lambda = \mathrm{e}^{2kh} > 1,$$

where the integral is along the real axis. It is noted that $\operatorname{Re} w(1) = 0$. Meanwhile $\operatorname{Im} w(1)$ may be considered as a function v of k and written

$$v(1;k) = k\log\frac{k}{\mathrm{e}} - \frac{\pi^2}{8h} + \frac{1}{2h}\int_{-1}^{1}\log(1+\lambda^{-1}u)\frac{\mathrm{d}u}{u}$$

so that $v(1;1) = -1 - \pi^2/8h$. However, because $k = \coth kh$, $h \to \infty$ as $k \to 1$ so this becomes $v(1;1) = -1$. Results of similar type were also written down by Friedrichs (1948).

By further differentiation of equation (3.2)

$$w''(\eta) = \mathrm{i}k\left(\frac{1}{\eta} + \frac{2kh}{\sinh 2kh\eta}\right)$$

so that, at $\eta = 1$, $k^{-2}w''(\eta)$, is proportional to the local group velocity of dispersive waves at depth h. This result appears to have gone unnoticed in Friedrichs (1948) although it corresponds to the function $j(\lambda)$ therein (in the notation of the present paper, $w''(1) = \mathrm{i}k^2 j(k^{-1})$) and as will be confirmed below relates also to the Burnside shoaling coefficient $D_d = [\tanh kh\,(1 + 2kh/\sinh 2kh)]^{-1/2}$ (Burnside 1914; Ehrenmark 1996).

Further differentiation yields $w'''(\eta) = -\mathrm{i}k(\eta^{-2} + 4k^2h^2 \cosh 2kh\eta \;\; \mathrm{cosech}^2 2kh\eta)$, giving $w'''(1) = -\mathrm{i}k(1 + h^2(k^4 - 1))$.

3.2. *The expansions*

Following the routine examples of e.g. (Copson 1965, p. 70) we can write $w(\eta) = w(1) - \tau$ where $\tau = \frac{1}{2}(\eta - 1)^2 f(\eta)$ and f has the expansion $f(\eta) = w''(1) + (\eta - 1)w'''(1) + O((\eta - 1)^2)$. Because $f(1) \neq 0$ there are just two steepest paths from the saddle point. At this point it is remarked that only one saddle (say that at $\eta = +1$) need be considered, for the original integrand in equation (2.3) is real on the real axis and so takes conjugate values at conjugate points. Accordingly, the usual saddle point argument then leads to the one-term expansion

$$\phi \sim \frac{kh}{\alpha}\mathrm{Re}\left\{ \mathrm{e}^{Xw(1)} \frac{\cosh k(z+h)}{\cosh kh} \left(\frac{2}{\mathrm{i}kX^2 w''(1)} \right)^{1/2} \int_0^\infty \tau^{-1/2}\mathrm{e}^{-\tau}\,\mathrm{d}\tau \right\}, \quad \alpha \to 0$$

which simplifies to

$$\phi \sim \sin(-v(1)X) \left(\frac{-2k\pi}{\mathrm{i}w''(1)} \right)^{1/2} \frac{\cosh k(z+h)}{\cosh kh}. \tag{3.4}$$

It is noted that $w''(1) \to \mathrm{i}k$ as $h \to \infty$ so the wave form has the correct amplitude $\surd(2\pi)$ there. The exact linear theory was considered in Ehrenmark (1996) and it was established there that a true shoaling coefficient was, for small beach slopes, very accurately modelled by the Burnside shoaling coefficient, here effectively proportional to $w''(1)^{-1/2}$, up to very close to the shore (i.e. provided h is not too small). This was also observed in Friedrichs (1948).

The phase variation $v(1)$ is of course the element of greatest interest in the present context. Friedrichs effectively treats h as a function of X. This approach is followed here, and a local wavenumber can then be defined by $\mathrm{d}(-hv(1))/\mathrm{d}h$. After some routine algebra this reduces identically to the value k, thus giving the surface wave proportional to $\sin kx$ and suggesting the validity of the formula $k = \coth kh$ at least for modest slopes.

In Ehrenmark & Williams (2001) the exact linear model was utilized to examine the accuracy of the response of Airy's dispersion equation when the slope was gradually increased. The findings generally were that the equation tended to underestimate the wavenumbers near the shore and that this deficiency increased substantially for steeper beaches. A finite difference computation for an initial value problem posed with the regular MSE was designed using both classical (Burnside) computations (k_1) and empirically enhanced values (k_2). This was compared with the exact solution and a suitable cumulative error norm computed. This showed that for k_1 relative error was about 25% for the 30° beach whilst for k_2 this was reduced to 9%. For the 45° beach these values were 28.6% and 5.5% respectively. Thus it became clear that, at least for that investigation (which was carried right up to the shoreline), significantly enhanced wavenumber values were required.

An attempt will now be made to extend Burnside's equation following Friedrichs' initial approach. To do this, higher-order expansions will be required and the saddle point method essentially replaced by the fuller method of steepest descent.

4. Method of steepest descent to order α

One restriction of the results in Friedrichs (1948) arises from the implicit assumption $X\alpha = h$. This restricts the subsequent asymptotic analysis to just a leading-term expansion. A more precise asymptotic investigation requires further terms in the expansion of the integrand. Thus take $X = h\cot\alpha$ instead and write $X\alpha = \mu h$ for convenience. While interest is focused on the surface wave behaviour, the term $\cos s(\theta+\alpha)/\cos s\alpha$ may be ignored. Expansions are required for $R^{-s}\Gamma(s)$ and for $B(s)\sin s\pi$. Correct to $O(\alpha)$,

$$B(s)\sin\pi s = \left(\frac{\pi}{2}\right)^{1/2}\exp\left\{\frac{\pi}{2\alpha}\log 2 + \alpha^{-1}\int_0^{\pi/2}\log\cos(s\alpha+\sigma)\,\mathrm{d}\sigma + \tfrac{1}{2}\log(-\cot s\alpha) + \frac{\alpha}{6\sin 2s\alpha}\right\}$$

whilst expression (3.1) can be retained as it is independent of α. The solution may then be expressed in the form

$$\phi = kX^{1/2}\mathrm{Re}\int_0^{\infty} q(\eta)\mathrm{e}^{Xw(\eta)}\,\mathrm{d}\eta \tag{4.1}$$

where

$$w(\eta) = \mathrm{i}k\eta\log\frac{\mathrm{i}k\eta}{\mathrm{e}} + \frac{1}{\mu h}\left(\frac{\pi}{2}\ln 2 + \int_0^{\pi/2}\log\cos(\mathrm{i}k\eta\mu h+\sigma)\,\mathrm{d}\sigma\right) \tag{4.2}$$

and

$$q = \exp\left\{\frac{1}{12\mathrm{i}k\eta X} - \frac{1}{2}\log k\eta\tanh\mu kh\eta + \frac{\alpha}{6\mathrm{i}\sinh 2\mu kh\eta}\right\}.$$

Differentiation of equation (4.2) gives $w'(\eta) = \mathrm{i}k\log(k\eta\tanh k\mu h\eta)$ and determines a stationary point at $\eta = 1$ if

$$k = \coth\mu kh \tag{4.3}$$

where $\mu = \alpha\cot\alpha$.

The expression for q then simplifies to

$$q = \exp\left(-\frac{1}{2\mathrm{i}k}w'(\eta) - \frac{1}{12Xk^2}w''(\eta)\right).$$

The question will naturally arise as to what extent equation (4.3) might represent an improvement to the classical dispersion relation when α is no longer very small. Before investigating this further, it is appropriate to determine more fully the asymptotic expansion of the wave using two terms of the method of steepest descent.

The value of $w(1)$ is readily extracted as before and can be written

$$w(1) = \mathrm{i}k\log\frac{\mathrm{i}k}{\mathrm{e}} - \frac{\mathrm{i}\pi^2}{8\mu h} + \frac{k\pi}{2} + \frac{\mathrm{i}}{2\mu h}\int_{-1}^{1}\log(1+\chi u)\frac{\mathrm{d}u}{u} \tag{4.4}$$

where $\chi = \exp(-2\mu kh)$. Again, with the integration taken on the real axis, it follows that $\mathrm{Re}\,w(1) = 0$. The revised values of w'', w''' and w^{iv} (denoted a, b and c below)

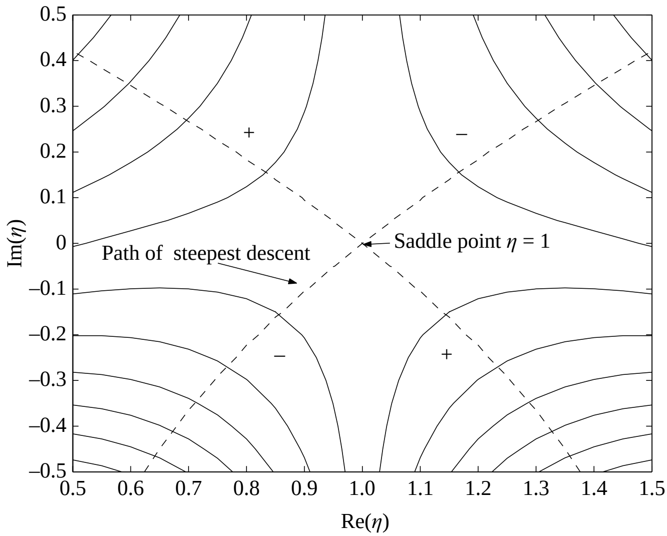

FIGURE 2. Contours of Re $w(\eta)$ and paths of steepest descent and ascent; case: $\alpha = \pi/4$; $h = 2$.

at $\eta = 1$ turn out to be respectively $\mathrm{i}k(1 + \mu h(k^2 - 1))$, $-\mathrm{i}k(1 + h^2\mu^2(k^4 - 1))$ and $2\mathrm{i}k(1 + h^3\mu^3(k^2 - 1)(k^4 + 1))$. Thus near the saddle point, set

$$w = w(1) - \tau, \ \tau = \frac{\mathrm{i}}{2}(\eta - 1)^2\left(|w''(1)| - \tfrac{1}{3}|w'''(1)|(\eta - 1) + \tfrac{1}{12}|w^{\mathrm{iv}}(1)|(\eta - 1)^2 \ldots\right).$$

The steepest paths are given by $\mathrm{Im}\,(w(\eta)) = \mathrm{Im}\,(w(1))$. These are computed (purely for illustration) from the full expression and shown in figure 2 for the particular case $\alpha = \pi/4$ taken at depth $h = 2$ where the solution of (4.3) gives the value $k = 1.072$ approximately (compared with the approximate value 1.033 from the (Burnside) Airy formula or 1.108 as computed in Ehrenmark & Williams (2001) with an *ad hoc* method for the plane beach; see figure 3 for wider comparison of these three methods). In Chamberlain & Porter (1999) the authors derive a number of useful approximations to the solution of the dispersion equation, thus obviating the need for iterative schemes. One of these approximations, in the present notation, becomes $k = (1 - \mu h \mathrm{e}^{-\mu h}/\sinh \mu h)^{-1/2}$ and yields the comparative value $k = 1.079$ at $h = 2$ for $\alpha = \pi/4$. Other approximations given are more accurate but also more cumbersome.

Near the saddle point, set $\eta = 1 + Z$ and $\tau = -\frac{1}{2}t^2$ and define the two branches of steepest descent by

$$Z_1 = a_1 t + a_2 t^2 + \ldots, \quad Z_2 = -a_1 t + a_2 t^2 + \cdots$$

from which it follows that

$$a_1 = \frac{1}{\sqrt{w''(1)}} \text{ and } a_2 = -\frac{1}{6}\frac{w'''(1)}{w''(1)^2}.$$

Also, on each branch, $q(\eta) = q(1) + Z_{1,2}q'(1) + O(Z^2)$ where the prime denotes a derivative with respect to η. Putting the results together and taking only the two dominant terms, it follows from equation (4.1) and the usual steepest descent

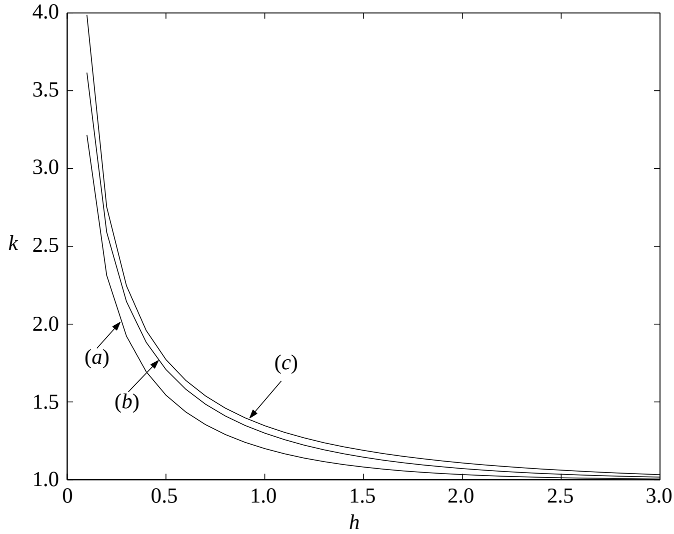

FIGURE 3. Dispersion curves from three different computations plotted against non-dimensional depth h: (a) solutions to $k = \coth kh$, (b) (present theory) solutions to $k = \coth \mu kh$, (c) as computed *ad hoc* in Ehrenmark & Williams (2001). Case: $\alpha = \pi/4$.

arguments that

$$\phi \sim \mathrm{Re}\,(-\mathrm{i}kX)^{1/2}\mathrm{e}^{\mathrm{i}Xv(1)}\int_0^\infty (q(1) + (Z_1 - Z_2)q'(1) + \cdots)\left(\frac{\mathrm{d}Z_1}{\mathrm{d}\tau} - \frac{\mathrm{d}Z_2}{\mathrm{d}\tau}\right)\mathrm{e}^{-\tau X}\,\mathrm{d}\tau, \quad (4.5)$$

as $X \to \infty$. That $\mathrm{d}Z_1/\mathrm{d}\tau - \mathrm{d}Z_2/\mathrm{d}\tau$ is bounded when $\tau \geqslant \tau_0$ for some $\tau_0 > 0$ is readily seen from the result

$$\frac{\mathrm{d}Z}{\mathrm{d}\tau} = \frac{-1}{w'(\eta)} = \frac{\mathrm{i}}{k\log\left(k\eta\dfrac{1-\chi^\eta}{1+\chi^\eta}\right)}.$$

The two-term asymptotic expansion can now be assembled from details in (Roseau 1976, pp. 108–111) subject to the validity of the required contour deformation. This is readily demonstrated because all singularities of the original integrand are on the real s-axis and the saddle point in question is on the positive imaginary s-axis at a distance $> X$ from the real axis. Moreover, the convergence of the integral as $\mathrm{Im}\,s \to \pm\infty$ is absolute in the left-hand half-plane $\mathrm{Re}\,s < \frac{1}{2}$ (see e.g. Ehrenmark 1987 for full details of asymptotics required). The expansion is therefore

$$\phi \sim \left(\frac{-2\pi k}{\mathrm{i}w''(1)}\right)^{1/2}\mathrm{Re}\,\mathrm{e}^{\mathrm{i}Xv(1)}\left(1 + \frac{\mathrm{i}\Omega}{2X} + \ldots\right)q(1), \quad X \to \infty \quad (4.6)$$

where

$$\mathrm{i}\Omega = \left[\frac{-q''}{qw''} - \frac{5}{12}\frac{(w''')^2}{(w'')^3} + \frac{1}{4}\frac{w^{iv}}{(w'')^2} + \frac{q'w'''}{q(w'')^2}\right]_{\eta=1}$$

and Ω is real to the leading order. When $\Omega = \mathrm{o}(X)$ equation (4.6) can be interpreted as

$$\phi \sim \left(\frac{-2\pi k}{\mathrm{i}w''(1)}\right)^{1/2}\cos\left(Xv(1) + \frac{\Omega_0}{2X}\right), \quad X \to \infty \quad (4.7)$$

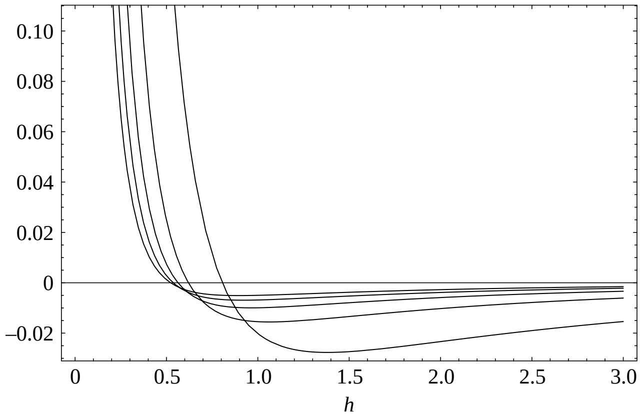

FIGURE 4. Relative values of the second asymptotic term: plots show $\Omega_0/(2X^2v(1))$ for beach angles $\alpha=\pi/3$ (rightmost curve), $\pi/4, \pi/5, \pi/6,$ and $\pi/7$ (leftmost curve) against non-dimensional depth h.

and the effective wavenumber k^* (see Appendix B) is then

$$k^* = -k + \frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{\Omega_0}{2X}\right).$$

Denoting $[a, b, c]$ as $[w'', w''', w^{\mathrm{iv}}]_{\eta=1}$, the dominant part $\mathrm{i}\Omega_0$ is found when $-q''/qa + q'b/qa^2 \sim a/4k^2$ and hence can be written

$$\mathrm{i}\Omega_0 \sim \frac{a}{12k^2} - \frac{5}{12}\frac{b^2}{a^3} + \frac{1}{4}\frac{c}{a^2}, \quad X \to \infty.$$

The validity of the expansion breaks down when $\Omega/2X = O(X)$. The quantity $\Omega_0/(2X^2v(1))$ has been computed against depth for a range of angles (see figure 4). In order to gauge more fully the significance of these results, presented in figure 5 is a more extensive graphical representation against beach angle where the curves show the depths of flow at which the second asymptotic term is respectively 5%, 10% and 20% of the first term. Although rather imprecise, this might serve as a good indicator to modellers of boundaries that should not be transgressed lightly.

5. Numerical tests

The two basic types of test to be considered are first tests on the classical plane beach problem for which exact solutions are well known and easily computed and secondly tests on the ramp profile of Booij for which 'exact' solutions are taken from Booij's full linear model (see Booij 1983) but here calculated (and supplied in a personal communication) by R. Porter using an integral equation approach.

5.1. *The beach problem tests*

Tests of MSE response to varying-wavenumber calculations in the flat-beach problem were designed and executed in Ehrenmark & Williams (2001). There, the original MSE (Berkhoff 1973), the Modified MSE (Chamberlain & Porter 1995) and the extended MSE (Massel 1993) were all tested using Airy values for k and also using the values obtained from consideration of the known exact solution. These tests all revealed that the response was much more accurate using the latter values, particularly for steeper beaches where it was clear that Airy theory would significantly underestimate

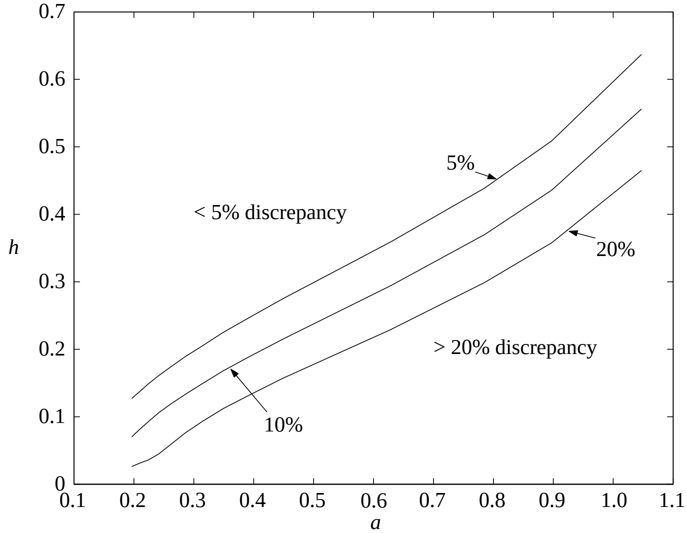

FIGURE 5. Boundaries of relative discrepancy between one- and two-term asymptotic expansions plotted against beach angle α and non-dimensional depth h. Contours show 5%, 10% and 20% discrepancy boundaries equivalent to $\Omega_0/(2X^2 v(1)) = [\,0.05, 0.1, 0.2]$.

the values of k (illustrated in the respective curves (a) and (c) in figure 3) for the case of the beach of unit gradient. Figure 2 shows also that the dispersion curve (b) calculated from the new formula $k = \coth \mu k h$ is much closer to the 'exact curve' (c) so that improved results can be expected.

Some of the tests of Ehrenmark & Williams (2001) are thus repeated here using revised values for k and compared with Airy values. The reader is referred to that work for the full details of the integration. Obviously the interest here is in steeper beaches, so the comparison is limited, for brevity, to two beaches namely $\alpha = \pi/6,\ \pi/4$. There is not a great deal of difference in the new performance of the modified MSE compared to that of the extended MSE so the results of the latter are not published here. It is emphasized again that this is not a test of the performance of various MSEs (there are other more improved versions now e.g. Porter 2003) but simply a test of how a given MSE might give better results using the new k values. It is found that both the original MSE and the modified MSE give significantly increased accuracy. Results for the modified MSE are given in figure 6 for respectively the regular and singular waves on the unit-gradient beach. A similar display for a beach of inclination 30° using instead the original MSE by Berkhoff (1973) is given in figure 7. Shown also, in each case, are the curves obtained with the theory in Ehrenmark & Williams (2001) which were developed on an *ad hoc* basis. In all cases there is clear evidence of a very significant increase in accuracy, particularly at very shallow depths, when the new dispersion equation $k = \coth \mu k h$ is used in place of $k = \coth kh$.

5.2. *The ramp-profile tests*

The normal-incidence ramp-profile test of Booij (1983) is recalculated (see figure 8) using first the basic MSE (Berkhoff 1973). Readers are referred to Booij (1983) for full details of the geometry. The object again is to examine the improvement that may be obtained by using the revised wavenumber values, so this basic MSE is used and improvements in results for steeps slopes are comparable to those noted in Chamberlain & Porter (1995) using the more accurate modified MSE with

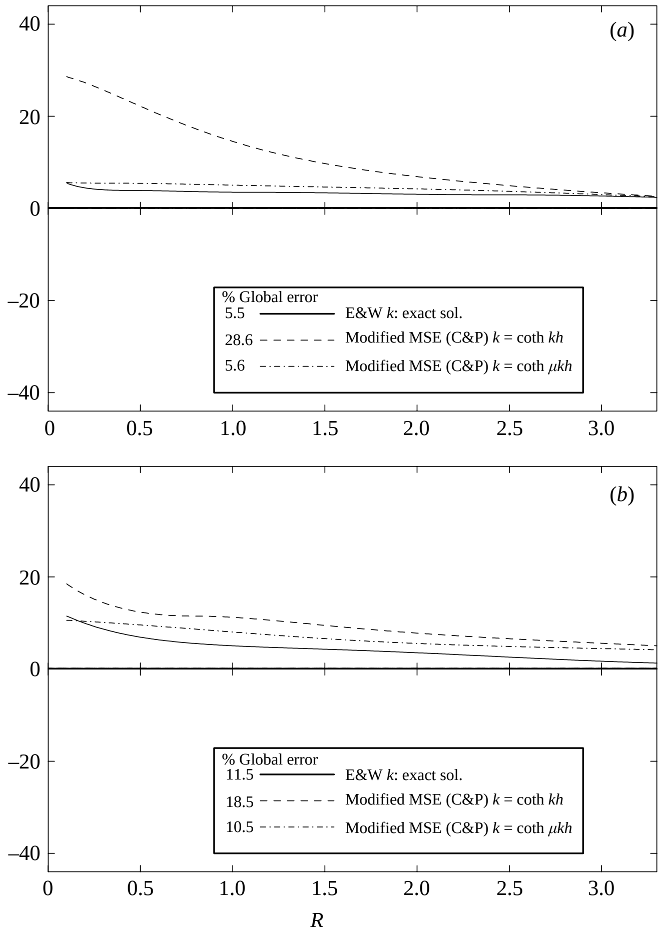

FIGURE 6. Initial value problem started at $R = 20$: Plots show % cumulative error in response by the modified MSE (Chamberlain & Porter 1995) to the old and new dispersion relation. Case: $\alpha = \pi/4$. (*a*) Regular wave; (*b*) singular wave. Results of Ehrenmark & Williams (2001) shown for comparison.

old wavenumbers. The question of the alternative mass continuity requirements at the points where h' is discontinuous, discussed by Porter & Staziker (1995) is not considered for this equation.

It is also applied to the modified MSE with mass conservation maintained by jump conditions where h' is discontinuous. This shows remarkable accuracy when the new dispersion relation is used. Figure 9 shows the nature of the improvement obtained for the steepest inclines. A value of 0.2263 is obtained for the reflection coefficient at $W_s = 0.05$ (steepness gradient 8) compared with 0.2276 computed from the full linear model and compared also with 0.2604 from the modified MSE (without evanescent waves) as in Porter & Staziker (1995). See also Appendix C for some details.

6. Concluding remarks

A revised equation for the general transformation of wavenumber under conditions of varying depth, namely

$$k = \coth \mu k h$$

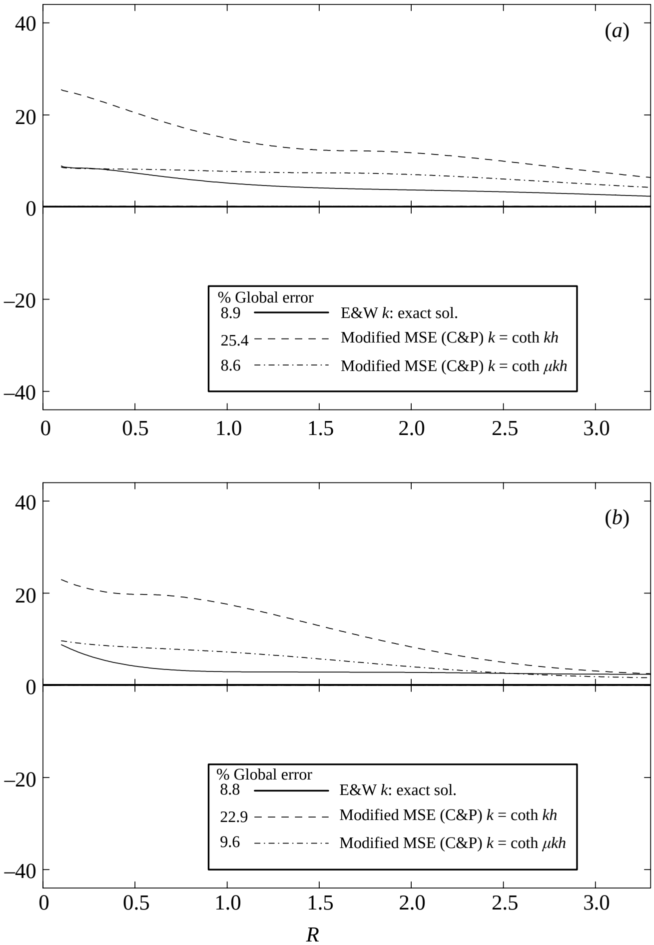

FIGURE 7. Comparison of response by original MSE (Berkhoff) to old and new dispersion relation. Case: $\alpha = \pi/6$. (*a*) Regular wave; (*b*) singular wave. Results of Ehrenmark & Williams (2001) shown for comparison.

where $\mu = \alpha \cot \alpha$, has been suggested from an asymptotic analysis of the exact solution of the linearized beach problem. This has correspondingly led to a revised estimate for group velocity on slopes of gradient $\tan \alpha$. The tests included here have been restricted to those on a plane beach and the standard test against the ramp profile used by Booij (1983) as this is now routinely used by most authors to examine the performance of their various versions of the MSE. There remains the interesting possibility of developing a numerical test using the exact solution of Roseau (1976, pp. 100–106) for waves over a specific curved bottom which recovers uniform depth as $X \to \infty$. This work is currently being undertaken as a PhD project and will be reported in due course. Also, a referee has raised the interesting question of extension to a three-dimensional analysis. This could be investigated, at least for oblique waves on a plane incline, using a similar model to the one herein with the (inverse) Mellin transform replaced by the Kontorovich–Lebedev transform (see e.g. Ehrenmark 1998).

One aspect of the present solution which is also discussed is the usefulness of a two-term expansion, such as that developed herein, in helping assess the range of

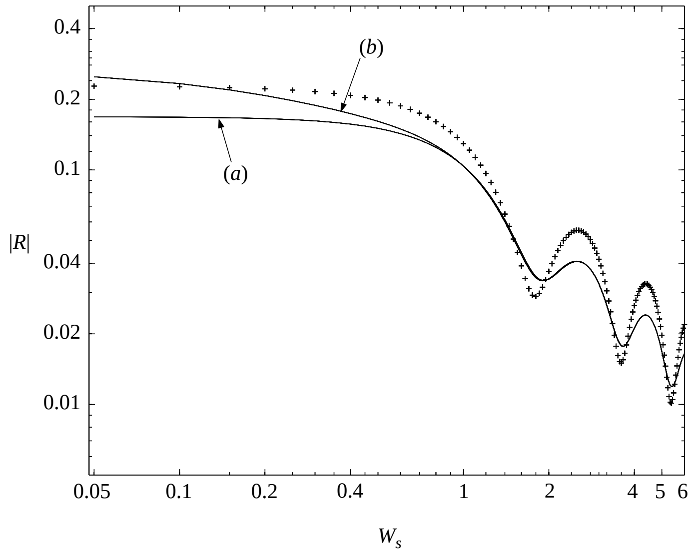

FIGURE 8. Comparison of reflection coefficient response $|R|$ on Booij's ramp by the original MSE (Berkhoff) to the old (curve (*a*)) and new (curve (*b*)) dispersion relation. Horizontal axis W_s gives non-dimensional horizontal extent of ramp; vertical extent is a fixed 0.4 units. The $+$ indicate the full linear solution.

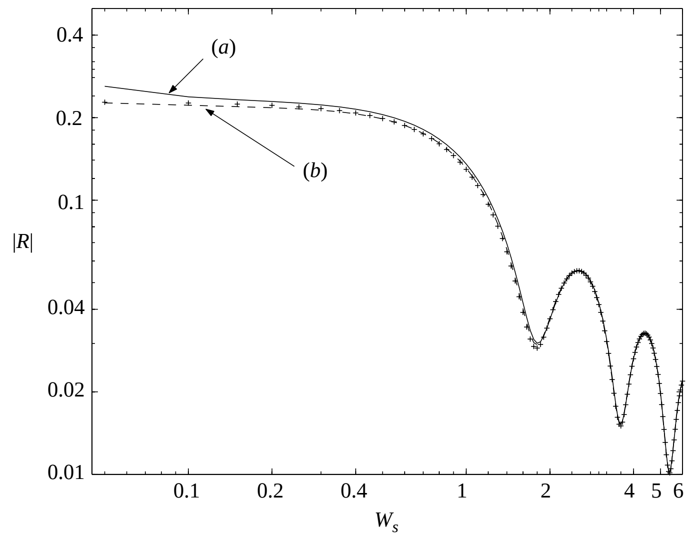

FIGURE 9. Comparison of reflection coefficient response $|R|$ on Booij's ramp by the modified MSE with mass conservation (Porter & Staziker) to the old (full line (*a*)) and new (broken line (*b*)) dispersion relation. Horizontal axis W_s gives non-dimensional horizontal extent of ramp; vertical extent is a fixed 0.4 units. The $+$ indicate the full linear solution.

validity of the one-term expansion. Previous authors have generally left this discussion at a qualitative stage, e.g. Burnside (1914) speaks of negligible reflection as a criterion whilst Friedrichs (1948, p. 114) refers to validity to within one third of a wavelength from the shore-line. Here, figures 4 and 5 indicate some more definitive boundaries

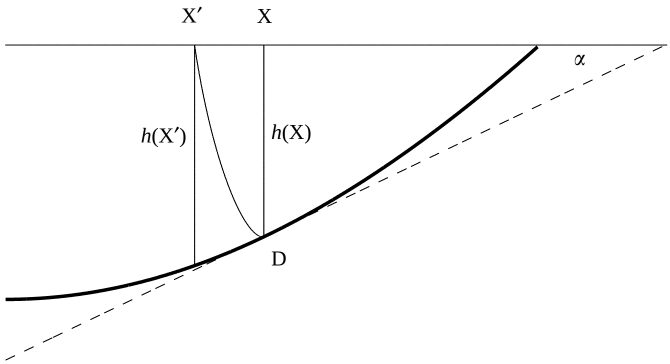

FIGURE 10. Schematic diagram, with X as a true field point and X′ as a virtual field point, relating actual depth h to virtual depth $h' = h(\mathrm{X}')$ inferred from Friedrichs' asymptotic analysis. Note that $h(\mathrm{X}) \simeq h(\mathrm{X}')\alpha \cot\alpha$ to $O(\alpha^2)$.

for the safe application of the dispersion relation. Interestingly, in his first experiment on the ramp profile, Booij (1983) found that in water of minimum non-dimensional depth 0.2, his results appear to become unreliable when the bottom slope exceeds 0.4. With reference to figure 5 one can identify this point precisely in the transitional band between 5% and 10% discrepancy, i.e. where one might expect the asymptotics to become somewhat questionable. The new dispersion relation evidently pushes this boundary to a shallower depth for fixed angle for the same basic version of MSE. More refined models do this anyway: as seen in Chamberlain & Porter (1995) the modified MSE (MMSE) can be used with reasonable accuracy up to gradient 4 and in Porter & Staziker (1995) (also MMSE but with mass conservation) even steeper gradients, whilst in Kim & Kwang (2004) the authors developed a complementary MSE based on stream function and appear to have kept the same order of accuracy as Booij for slopes of order unity. There is no inference that most of the errors incurred by Booij and others for the very steep ramp are due only to the differences between the old and new dispersion relation. Indeed, Porter & Staziker (1995) have, to some extent, accounted for Booij's inaccuracy at steep slopes through the neglect of mass conservation in the model. On the other hand it has been seen that use of the new dispersion relation with Berkhoff's original MSE has a similar effect and that for the modified MSE the results for steeper beaches are almost indistinguishable from those of the full linear model.

A simple physical explanation for why the new dispersion relation should be used can be inferred from figure 10. This shows a true field point X associated with a point D on a curved bottom. Friedrichs' analysis is based on the asymptotic approximation $X \sim h/\alpha,\ X \to \infty$ and so this same point D will in turn be associated with a virtual field point X' on the surface found by drawing the arc centred where the tangent at D cuts the free surface. Thus the virtual depth h' which would be used in the classical dispersion relation is overestimated as shown and this results in the underestimate of the wavenumber k. Clearly the ratio between h and arc length is $\alpha \cot\alpha$.

The author is indebted to Dr Richard Porter (University of Bristol) for kindly providing the 'exact' comparative data of the ramp problem derived by an integral equation approach. Additionally, a number of helpful suggestions by the referees have contributed to tidy up the manuscript.

Appendix A. Asymptotic analysis with arbitrary α

In this Appendix the relaxation of the special values of slope that were considered in Friedrichs (1948), and here in the main text, is discussed in order to simplify the asymptotic analysis. Thus an alternative asymptotic analysis only for $B(s)$ is required for the value $s = \mathrm{i}k\eta X$ where $X \gg 1$. The additional notation $B_m(s)$ is introduced to indicate the dependence of B on the nature of α where now $\alpha = \pi/2m$ and $m \geqslant 1$ but not necessarily integer. We must adopt the more general expression (Ehrenmark 1987)

$$B_m(s) = \Gamma(s)\exp\left[\int_0^\infty \frac{\mathrm{d}t}{t}\left\{\frac{2\mathrm{e}^{t/2}\sinh(s-\frac{1}{2})t}{(\mathrm{e}^{mt}+1)(e^t-1)} - \left(s-\tfrac{1}{2}\right)\mathrm{e}^{-t}\right\}\right], -m < \operatorname{Re} s < m+1. \tag{A 1}$$

Using equation (A 1) and Kummer's result (Whittaker & Watson 1952, p. 250), B_m can be expressed in the alternative form

$$B_m(s) = \left(\frac{\pi}{\sin \pi s}\right)^{1/2} \exp\left\{-\frac{1}{2}\int_0^\infty \frac{\sinh(2s-1)t}{\sinh t}\frac{\tanh mt}{t}\mathrm{d}t\right\}, \quad 0 \leqslant \operatorname{Re}(s) \leqslant 1.$$

Let $2N-1 \leqslant m < 2N+1$. Note that equation (2.4) follows trivially from either of these integral expressions. The process requires evidence that the asymptotic expression for B_m is identical to that for B_{2N+1} in the dominant term. Since the dominant exponent was earlier seen to be $O(m)$ it must now be shown that the terms in the exponent for B_m/B_{2N+1} are all $o(m)$.

The integral above (denoted I) may be considered in two parts, i.e. $I = I_1 - I_2$ where

$$I_1(m) = \int_0^\infty \frac{\sinh 2st}{t}\frac{\tanh mt}{\tanh t}\,\mathrm{d}t,$$

and

$$I_2(m) = \int_0^\infty \frac{\cosh 2st}{t}\tanh mt\,\mathrm{d}t,$$

an integral which is readily evaluated when $s = \mathrm{i}\mu m$ and where now $\mu = kh\eta/(m\tan\pi/2m)$. From Art. 7.116 of Ditkin & Prudnikov (1965) it follows that

$$I_2 = \log\coth\frac{\pi\mu}{2} = O(1).$$

Put $M = 2N+1$ for convenience. The estimation of $J = I_1(m) - I_1(M)$ is now required for $m \gg 1$. Clearly

$$J = \int_0^\infty \frac{\sin 2\mu mt}{t}\frac{(\tanh mt - \tanh Mt)}{\tanh t}\,\mathrm{d}t + \int_0^\infty \frac{(\sin 2\mu mt - \sin 2\mu Mt)}{t}\frac{\tanh Mt}{\tanh t}\,\mathrm{d}t.$$

The first of these integrals, J_0, is

$$\int_0^\infty \frac{\sin 2\mu mt}{t}\frac{\sinh(m-M)t}{\sinh t}\frac{\cosh t}{\cosh Mt\cosh mt}\,\mathrm{d}t$$

and since $m - M < 1$ and $\sin 2\mu mt < 2\mu mt$ the estimate

$$|J_0| < 2\mu m\int_0^\infty \frac{1}{\cosh Mt}\,\mathrm{d}t = O(1)$$

For the second integral $2J_1$, follows.

$$J_1 = \int_0^\infty \frac{\cos\mu(m+M)t\sin\mu(m-M)t}{t}\left(\frac{\tan hMt}{\tanh t}-1\right)\mathrm{d}t + \int_0^\infty \frac{\cos\mu(m+M)t\sin\mu(m-M)t}{t}\,\mathrm{d}t.$$

Here the second of these integrals is $O(m^{-1})$ by the Riemann–Lebesgue lemma whilst the first one (J_{10}) can be estimated by splitting the integration interval. Note first that $|\tanh Mt/\tanh t - 1| < M-1$ so that, with $(\cdot)$ denoting the integrand of J_{10},

$$\left|\int_0^{1/M}(\cdot)\,\mathrm{d}t\right| < (1-1/M)(M-m)\mu = O(1).$$

For the remaining part, write $(\tanh Mt - \tanh t) = \sinh(M-1)t/(\cosh\ Mt\ \cosh t)$ so that

$$\left|\int_{1/M}^\infty(\cdot)\,\mathrm{d}t\right| < \mu(M-m)\int_{1/M}^\infty \frac{\sinh(M-1)t}{\cosh Mt\sinh t}\,\mathrm{d}t = O(1).$$

Hence the terms at $O(m)$ are unaffected in the analysis. Therefore the saddle point analysis will remain as it was in the case of the special slope angles.

Appendix B. Proof of wavenumber

For a wave variation $\cos Xv(1)$ the wavenumber can be defined by $(\mathrm{d}/\mathrm{d}X)\,Xv(1)$ and it is shown here that this quantity is identically equal to k when $k = \coth\mu kh$. From the main text,

$$v(1) = \operatorname{Im} w(1) = k\log k/\mathrm{e} - \pi^2/8H + \frac{1}{2H}\int_{-1}^1 \log(1+\mathrm{e}^{-2kH})\frac{\mathrm{d}u}{u},$$

with $H = \mu h$ for convenience. On differentiation (denoted by$'$)

$$\frac{\mathrm{d}}{\mathrm{d}H}(Hv(1)) = k\log k/\mathrm{e} + Hk'\log k - (kH)'\mathrm{e}^{-2kH}\int_{-1}^1 \frac{1}{1+u\mathrm{e}^{-2kH}}\,\mathrm{d}u.$$

Completing the integral, the last term is equivalent to $-(kH)'\log\coth kH$. In view of the dispersion relation itself, it therefore follows that

$$\frac{\mathrm{d}}{\mathrm{d}H}(Hv(1)) = \frac{\mathrm{d}}{\mathrm{d}X}(Xv(1)) = -k$$

as required.

Appendix C. Ramifications of use with the modified MSE

The use of the modified mild-slope equation (MMSE) is not entirely straightforward. The functions u_0, u_1 and u_2 developed in Chamberlain & Porter (1995) are based on the nature of the choice of profile function w_0 in the approximation $\phi(x, y, z) = \phi_0(x, y)w_0(x, y, z)$ where z is the vertical coordinate and w_0 is assumed slowly varying in x. The function

$$w_0 = \frac{\cosh\mu kh\left(1+\dfrac{\tan^{-1}(z/x)}{\tan^{-1}(h/x)}\right)}{\cosh\mu kh}$$

is a choice which ensures that the surface boundary condition is satisfied exactly whilst at the bed $\partial w_0/\partial z = 0$. Thus, in common with the frequently used w_0, namely $\cosh k(z+h)\mathrm{sech} kh$, the bottom condition is satisfied exactly by w_0 only on horizontal bottoms. Moreover, in common with the approach in the asymptotic analysis, only the dominant terms (for small α) need be considered. It is easily established, with reference to equations (2.11) and (2.16) in Porter & Staziker (1995), that this amounts essentially to u_0 being replaced by u_0/μ, u_1 being invariant and u_2 replaced by μu_2 once k has been determined by $k = \coth \mu kh$. These changes were incorporated in the calculations shown in figure 9.

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